PDE - based Image Processing

– what is it and what can it be used for?

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A collaboration between:

Dept. for Optical measurement systems and data analysis
Dept. for Applied Mathematics
Outline

- PDE (Partial Differential Equations)
  - What is this in image processing?
  - How is it different in PDE-based image processing?

- Anisotropic filtering

- Unsupervised segmentation

- Supervised segmentation

- What did we learn?
Diffusion process described with PDE versus Gaussian low pass filtering

- **PDE-based approach**
  - We want to minimize the image variation:

  \[
  \min_{I: \Omega \to \mathbb{R}} E(I) = \int_{\Omega} \| \nabla I \|^2 \, d\Omega
  \]

  Leads to heat equation ...

  \[
  \begin{cases}
  I(t=0) = I_0 \\
  \frac{\partial I}{\partial t} = \text{div}(\nabla I)
  \end{cases}
  \]

- **Image processing approach**
  - We want to reduce random high frequency noise, with minimum diffusion of image:

  - Apply a Gaussian low pass filter

**Different approach, equivalent result**
Why is PDE-based image processing different?

- Developed by mathematicians
  - Different terminology
  - Different way of thinking and different approach to problem
  - Different toolbox of basic methods

- This results in
  - Different results
  - Different problems solved
Different thinking and approach

<table>
<thead>
<tr>
<th>Image processing community</th>
<th>Mathematical community</th>
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<tbody>
<tr>
<td>Images are 2 or 3D matrices of pixel values</td>
<td>Images are continuous scalar or vector field</td>
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<tr>
<td>Operators are defined discretely, often as logical or matrix operations on pixel regions</td>
<td>Operators are developed and defined in the continuous space as a process described by PDEs</td>
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<td>Image filtering is done by convolution filters, linear or nonlinear</td>
<td>Image filtering is done by diffusion processes, linear or non-linear</td>
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<td>Image analysis is done by heuristic algorithms, often searching for contours and specific features</td>
<td>Image analysis is done by defining a functional that is to be optimised</td>
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- The mathematical approach gives a higher level of abstraction, but is less specific for the problem
Anisotropic filtering formulated as a diffusion process
Example: Facelift

I look terrible. What can I do?

Use Gauss filter? No, too blurred!

Repeated median filter. Maybe?

Anisotropic PDE filter? Yes!!

By Tom Kavli, SINTEF
Anisotropic filtering – what is the trick?

- Make a map of the gradient directions
- Make a diffusion tensor $D$ with its largest eigenvector normal to the gradient
  - Use the diffusion tensor to direct the smoothing mostly along edges, little across edges

\[
\frac{\partial I}{\partial t} = \text{div}(\nabla I) \quad \Rightarrow \quad \frac{\partial I}{\partial t} = \text{div}(D(x,y)\nabla I)
\]

Recommended reference: Thomas Brox; http://www-cvpr.iai.uni-bonn.de/brox/
Filtering of a seismic fault image, example 1

A 2D intersection of the 3D volume of seismic data

After anisotropic filtering

After anisotropic shock filtering

By Tom Kavli and Sigmund Clausen, SINTEF
Filtering of a seismic fault image, example 2

A 2D intersection of the 3D volume of seismic data

By Tom Kavli and Sigmund Clausen, SINTEF
Taking anisotropic filtering to the extreme
Shock filter: smoothing along texture, sharpening normal to texture

Recommended reference: Joachim Weickert;
http://www.mia.uni-saarland.de/weickert/index.shtml

By Sigmund Clausen, SINTEF
Smoothing formulated by a cost function to be minimized
- Color image inpainting

\[ E(I) = \int_{\Omega} \varphi(\| \nabla I \|) d\Omega + \lambda \int_{\Omega} \psi(I - U) d\Omega \]

- The inpainting area is defined prior to smoothing.
- No attachment to the input image (U) within inpainting area (\( \lambda \) is zero).
- Full attachment outside inpainting area (\( \lambda \) is infinity).

By Odd Andersen, SINTEF
Unsupervised segmentation
Unsupervised segmentation
- formulated as a cost functional to be minimized

\[ E(\Gamma) = -\int_{\Omega_1} \log p_1(I(x), x) dx - \int_{\Omega_2} \log p_2(I(x), x) dx + \nu \int_{\Gamma} ds \]

Recommended reference: David Tschumperlé; http://www.greyc.ensicaen.fr/~dtschump/
Unsupervised segmentation
- formulated as a functional to be optimized

\[ E(\Gamma) = -\int_{\Omega_1} \log p_1(I(x), x) dx - \int_{\Omega_2} \log p_2(I(x), x) dx + \nu \int_{\Gamma} ds \]

- Uses brightness, colour and texture as input attributes to the probability models
- Uses level set formulation for implicit representation of regions
- Simple initial probability models are iteratively updated based on intermediate segmentations
- Maximizes segmentation probability while maintaining simple shape
Improved probability models:

- Normal distribution can fail to model a region adequately.
  - Example: the black and white stripes of a zebra

- If the region is *sufficiently large*, we can use a smoothed histogram as a probability distribution.
  - This greatly improves segmentation on this kind of images
Multi-scale algorithms makes segmentation more robust and faster

- **Init. partition**
- **Level 0**
- **Level 1**
- **Level 2**

- Smooth heavily at coarse resolutions, less at finer resolution where we already have a good initial starting point
- Simple statistics on small levels, advanced statistics on large levels
Adding prior statistical models for object color and texture

Guiding the algorithm to find the desired objects
A 3D segmentation of the leg muscle from an MR image

- By specifying a prior statistical model for the muscle tissue the algorithm is guided to segment the muscles, and not other tissues in the image.
- Done by SINTEF within the European network AIM@SHAPE (http://www.aimatshape.net/).

By Odd Andersen, SINTEF
Adding shape models to guide the segmentation
Including prior shape knowledge in functional:
Daniel Cremers; http://www-cvpr.iai.uni-bonn.de/cremers/

PCA of hand shapes

Shape energy defined from the first p.c.'s

Total energy

\[ E(z) = E_{image}(u, C_z) + \alpha E_{shape}(\bar{z}) \]

High energy if the shape deviates from the statistical normal shapes

Low energy if the contour follows gradients in the image, high energy otherwise
Cremers demonstrates good segmentation with relatively cluttered background.
Test case: Segmentation of fish

- **Challenges**
  - Many overlapping fish =>
    - Partly occluded
    - No difference between foreground and background
    - Low contrast edges
  - Varying lighting and appearance

- **Opportunities**
  - Little variation in shape
  - Stereo images
  - Time series of images
Cremers’ method applied for fish

Initialization

Without prior shape model

With prior shape model
Cremers’ method trapped in a local minimum

Only one fish correctly identified
With a global search for minimum cost

The two best candidates are shown for each case.
Several local maxima causes problems for the PDE optimization. Result depends very much on the initialization.

Highest score

Second highest score
Summary

- PDE based image processing represents an interesting addition to “common” image processing
- Anisotropic filtering is very useful and works well
- Segmentation by optimization of a cost functional requires
  - Good prior models should be used for color, texture and/or shape to guide the segmentation to the desired results
  - The general models used by most researchers are often not enough restricted or they do not fit the real world well enough
  - There are often severe problems with local minima
  - There is a fast ongoing development of new methods and algorithms